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L^p NORMS OF CERTAIN KERNELS ON THE N-DIMENSIONAL TORUS

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ABSTRACT. In this paper we study a class of kernels F_R which generalize the Bochner-Riesz kernels on the N-dimensional torus. Our main result consists in upper estimates for the L^p norms of F_R as R tends to infinity. As a consequence we prove a convergence theorem for means of functions belonging to suitable Besov spaces.

1. Throughout this paper we identify the N-dimensional torus T^N ($N \geqslant 2$) with the N-dimensional cube $Q_N = \{x \in R^N\colon -1/2 < x_j \leqslant 1/2\}$. $S \subseteq R^N$ will denote a bounded open set. Suppose f is a complex-valued function defined on R^N which vanishes outside S and is continuous on S. For every $g \in L^1(T^N)$ and $R > 0$ we form the means

$$F_R * g(t) = \sum_m f(R^{-1}m)\hat{g}(m)\exp(2\pi imt) \tag{1}$$

where $t \in T^N$, m ranges on the integer lattice Z^N, and $*$ denotes the convolution in T^N.

In several cases one is able to estimate the $L^1(T^N)$ norm of the kernel

$$F_R(t) = \sum_m f(R^{-1}m)\exp(2\pi imt) \tag{2}$$

and (if $0 \in S$ and $f(0) = 1$) deduce convergence results for the means (1) (as $R \to \infty$) when g belongs to certain classes of periodic functions (see e.g. **[1]**–**[3]**, **[7]**, **[8]**, **[12]**). For instance, if $f(x) = (1 - |x|^2)^\delta$ when $|x| < 1$ and $f(x) = 0$ when $|x| \geqslant 1$, then $F_R = K_R^\delta$ is the familiar Bochner-Riesz kernel. In this case K. I. Babenko (cf. **[1]**) has shown that $\|K_R\|_1$ is of the same order as $R^{(N-1)/2-\delta}$ as R tends to infinity ($N \geqslant 2$, $0 \leqslant \delta < (N-1)/2$), while Stein proved the estimate $\|K_R\|_1 \sim \log R$ if $\delta = (N-1)/2$ **[9]**. Recently Yudin **[12]** obtained the estimate $\|F_R\|_1 = O(R^{(N-1)/2})$ if f is the characteristic function of a closed balanced set C, whose boundary has finite upper Minkowski measure. The method of Yudin (which also yields estimates for the $L^p(T^N)$ norms of F_R) uses Jackson approximation theorem, which in turn involves estimates for the $L^2(R^N)$ modulus of continuity of f. Such a method can be adapted to a more general situation, as we show in this paper. In §§2–4 we consider kernels of the type (2) associated with functions f whose derivatives of certain orders are controlled by (not necessarily positive)

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powers of the distance from the boundary of S. We prove an estimate from above for the $L^p(T^N)$ norms of F_R, which reduces to the estimate (from above) proved in **[1]**, **[8]**, **[9]**, **[12]** when F_R coincides with one of the kernels studied in these papers. (It should be remarked, however, that the authors of the first paper also obtain results for much more general kernels not expressed by means of trigonometric polynomials.)

In §5, as an application of the foregoing results, we prove a convergence theorem for the means (1) when g belongs to suitable Besov spaces (see Nikolskii **[6]** and Taibleson **[11]** for definition and properties of Besov spaces of periodic functions).

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2. In this section we fix the notation and the assumptions that will be used in the following. Nikolskii classes of periodic functions will be denoted by $H_p^k(T^N)$. We recall that $H_p^k(T^N)$ is defined as the Besov space of all the periodic functions g such that: (i) g and all its (generalized) derivatives of order r ($r + \tau = k$, r a nonnegative integer, $0 < \tau \leqslant 1$) belong to $L^p(T^N)$, (ii) if $D^\xi g$ is such a derivative, then $\|\Delta_h^2(D^\xi g)\|_p \leqslant \text{const}|h|^\tau$, where $|\xi| = r$ and Δ_h^2 denotes the second difference relative to the increment h. Here we have adopted the usual conventions for multi-index ξ; we notice that the same symbol $|\cdot|$ will be used to denote euclidean norms (or moduli) and length of multi-index. The nonperiodic Nikolskii classes have an analogous definition. Such classes will be denoted by $H_p^k(R^N)$.

The $L^p(T^N)$ norms ($1 \leqslant p \leqslant \infty$) are denoted by $\|\cdot\|_p$, while for $L^p(R^N)$ norms we write $\|\cdot\|_{L^p}$. The symbol $*$ is used indifferently to denote convolution of functions on T^N or of functions on R^N. We specify which case we are dealing with only when confusion can arise.

If M is a subset of R^N, we shall denote by $\overline{M}$ its closure and by ∂M its boundary. Suppose $\Omega \subseteq R^N$ is an open set and r a nonnegative integer (or $r = +\infty$). As usual $C^r(\Omega)$ denotes the class of all continuous functions on Ω which are continuously differentiable up to the order r. Finally A will denote a constant which may vary from line to line.

Suppose $S \subseteq R^N$ is an open bounded set whose boundary S has finite upper Minkowski measure, i.e. $\limsup_{\varepsilon \to 0} \mu(\Gamma_\varepsilon)/\varepsilon < \infty$, where Γ_ε is the union of all the balls of radius ε centered at the points of ∂S and μ denotes the Lebesgue measure. In the following we will consider complex-valued bounded functions f on R^N satisfying the following assumptions:

$$f(x) = 0 \quad \text{if } x \text{ does not belong to } S; \tag{3}$$

there exist an integer $n \geqslant 0$ and real numbers $\alpha > -1/2$ and $\beta > -3/2$ such that

$$f \in C^{n+1}(S); \tag{4}$$

$$|D^\xi f(x)| \leqslant A d(x, \partial S)^\alpha \quad \text{if } |\xi| = n \text{ and } x \in S; \tag{5}$$

$$|D^\xi f(x)| \leqslant A d(x, \partial S)^\beta \quad \text{if } |\xi| = n + 1 \text{ and } x \in S \tag{6}$$

(where $d(x, \partial S)$ denotes the distance of x from the boundary of S).

We require a further condition. If (4)–(6) are satisfied with $n \geqslant 1$, f must also satisfy

$$f \in C^{n-1}(R^N). \tag{7}$$

Since f is supposed bounded, in the sequel we may assume $\alpha \geqslant 0$ whenever $n = 0$.

We note the following obvious consequence of the above assumptions.

LEMMA 1. *Suppose $S \subseteq R^N$ is a bounded open set such that S has finite upper Minkowski measure and f is a bounded complex-valued function on R^N satisfying* (3)–(7). *Then there exists a constant $A > 0$ such that*

$$|f(x)| \leqslant \mathrm{A}d(x, \partial S)^{\alpha+n} \quad \textit{for all } x \in S. \tag{8}$$

PROOF. (8) is obvious if $n = 0$. Otherwise, given $x \in S$, let $y \in S$ be such that $|y - x| = d(x, \partial S)$. Then, if $h = y - x$, $|\xi| = n - 1$,

$$|D^\xi f(x)| = \left| \int_0^1 \frac{d}{dt}(D^\xi f(y + th))\, dt \right| \leqslant A \int_0^1 d(x, y + th)^\alpha\, dt \leqslant A|h|^{\alpha+1}$$

whence (8).

We give a few examples in order to clarify the sense of the above assumptions on f and S.

EXAMPLE 1. Let S be an open bounded set whose boundary has finite upper Minkowski measure. If f is the characteristic function of S, then f satisfies (3)–(6) with $n = \alpha = \beta = 0$. This example corresponds to the case considered by Yudin **[12]**.

EXAMPLE 2. Suppose $E(x)$ is an elliptic homogeneous polynomial in N variables of degree r. Set $S = \{x \in R^N\colon E(x) < 1\}$. It is clear that ∂S has finite upper Minkowski measure. Set, for every real δ and η, $\delta \geqslant 0$,

$$\begin{aligned} f_{\delta,\eta}(x) &= (1 - E(x))^{\delta + i\eta} && \text{if } x \in S, \\ f_{\delta,\eta}(x) &= 0 && \text{otherwise.} \end{aligned}$$

First suppose $\eta = 0$ and write $\delta = r + \tau$, with r an integer, $r \geqslant 0$, $-1/2 < \tau \leqslant 1/2$. Then $f_{\delta,0}$ satisfies (3)–(7) with $\delta = n = r$, $\alpha = 0$, $\beta = 0$ if $\tau = 0$. If $\tau \neq 0$ then $n = r$, $\alpha = \tau$, $\beta = \tau - 1$. In particular, if $\delta = (N - 1)/2$ we have $n = (N - 1)/2$, $\alpha = 0$, $\beta = 0$ for N odd, while, for N even, n is equal to the integer part of $(N - 1)/2$, $\alpha = 1/2$, $\beta = -1/2$.

Suppose now $\eta \neq 0$, and, as before, $\delta = r + \tau$. If $\tau = 0$ (3)–(7) are satisfied for $\delta = r = n$, $\alpha = 0$, $\beta = -1$. If $\tau \neq 0$ then $n = r$, $\alpha = \tau$, $\beta = \tau - 1$.

It is clear from these examples that the kernels (2), with f and S satisfying our assumptions, generalize to 'arbitrary' open sets the spherical Bochner-Riesz kernels.

For every $u \in L^2(R^N)$ denote by $\omega_2(u, t)$ the $L^2(R^N)$ modules of continuity of u, i.e. $\omega_2(u, t) = \sup_{|h| \leqslant t} \|\Delta_h u\|_{L^2}$, where $\Delta_h u(x) = u(x + h) - u(x)$.

LEMMA 2. *Suppose S is an open set with compact closure such that ∂S has finite upper Minkowski measure. Let f denote a complex-value function on R^N satisfying*

(3)–(7). *Then* $D^\xi f \in L^2(R^N)$ *for every* ξ, *with* $|\xi| = n$. *Moreover, if we make the further assumption that* $\beta \neq -1/2$ *when* $\alpha \geqslant 1/2$, *then*

$$\omega_2(D^\xi f, t) \leqslant At^\gamma \tag{9}$$

where

$$\gamma = \min(1, \alpha + 1/2, \beta + 3/2). \tag{10}$$

PROOF. Let (here and in the following) $S_t = \{x \in S: d(x, \partial S) > t\}$. If $\alpha > 0$, it is clear that $D^\xi f \in L^2(R^N)$. Suppose $\alpha < 0$. Then $I_t = \int_{S_t} |D^\xi f(x)|^2\, dx$ is majorized by $\int_{S_t} d(x, S)^{2\alpha}\, dx$. Denote by $m(y)$ the distribution function of $\chi_{S_t}(x) d(x, \partial S)^{2\alpha}$ (where χ denotes characteristic function). Clearly $\|m\|_\infty = \mu(S_t) < \mu(S)$ and $m(y) = 0$ if $y > t^{2\alpha}$. Taking into account the assumption on ∂S, we have

$$m(y) = \mu\{x \in S_t: d(x, \partial S)^{2\alpha} > y\} \leqslant Ay^{1/2\alpha}.$$

Therefore

$$I_t = \int_{S_t} d(x, \partial S)^{2\alpha}\, dx < \int_0^1 \|m\|_\infty\, dy + A\int_1^{t^{2\alpha}} y^{1/2\alpha}\, dy \leqslant A(1 + t^{2\alpha+1}).$$

Letting t tend to 0 we obtain the first part of the lemma. Assume now that $\beta \neq -1/2$ if $\alpha \geqslant 1/2$. Let $|h| \leqslant t$. Then we easily get

$$\|\Delta_h D^\xi f\|_{L^2} \leqslant \|\chi_{S_{2t}} \Delta_h D^\xi f\|_{L^2} + 2\|\chi_{S\setminus S_{3t}} D^\xi f\|_{L^2} = I_1 + I_2.$$

We have from (6) and the mean value theorem

$$I_1^2 \leqslant At^2 \int_{S_{2t}} d(x + \theta_x h, \partial S)^{2\beta}\, dx, \qquad 0 < \theta_x < 1,$$

and from (5)

$$I_2^2 \leqslant A\int_{S\setminus S_{3t}} d(x, \partial S)^{2\alpha}\, dx.$$

First we consider the case $\alpha < 0$, $\beta < 0$. Denote by $q(y)$ the distribution function of $\chi_{S_{2t}}(x) d(x + \theta_x h, \partial S)^{2\beta}$. Clearly $\|q\|_\infty \leqslant \mu(S)$ and $q(y) = 0$ if $y > t^{2\beta}$. On the other hand the condition on ∂S implies $q(y) < Ay^{1/2\beta}$ for $1 < y < t^{2\beta}$, so that, arguing as in the proof of the first part of the lemma, $I_1^2 < At^2(1 + t^{2\beta+1})$. Moreover, similar considerations give

$$I_2^2 \leqslant A\int_{(3t)^{2\alpha}}^{+\infty} y^{1/2\alpha}\, dy < At^{2\alpha+1}. \tag{11}$$

Hence we have

$$\omega_2(D^\xi f, t) \leqslant A(t + t^{\alpha+1/2} + t^{\beta+3/2}) \leqslant At^\gamma \tag{12}$$

where γ is given by (10).

Suppose now $\beta < 0$, $\alpha \geqslant 0$ ($\alpha < 1/2$ when $\beta = -1/2$). Then the same computations as before give

$$I_1^2 \leqslant At^2(1 + t^{2\beta+1}) \quad \text{or} \quad I_1^2 < At^2(1 + |\log t|) \quad \text{if } \beta = -1/2.$$

Now let m denote the distribution function of $\chi_{S\setminus S_{3t}}(x) d(x, \partial S)^{2\alpha}$. It is clear that

$$m(y) = 0 \quad \text{if } y > (3t)^{2\alpha}. \tag{13}$$

On the other hand, for $y \leqslant (3t)^{2\alpha}, y \geqslant 0$,

$$m(y) \leqslant \mu\{x \in S: d(x, \partial S) \leqslant 3t\} \leqslant At. \tag{14}$$

Hence, by (13) and (14)

$$I_2^2 \leqslant A \int_0^{(3t)^{2\alpha}} t\, dy \leqslant At^{2\alpha+1}.$$

Concluding

$$\omega_2(D^\xi f, t) \leqslant A(t + t^{\beta+3/2} + t^{\alpha+1/2}) \tag{15}$$

if $\beta \neq -1/2$, while

$$\omega_2(D^\xi f, t) \leqslant A(t + t\,|\log t|^{1/2} + t^{\alpha+1/2}) \leqslant A(t + t^{\alpha+1/2}) \tag{16}$$

if $\beta = -1/2$, $0 \leqslant \alpha < 1/2$, $\alpha \geqslant 1/2$. Similar arguments establish the lemma in the other cases.

In the next section we shall exclude the case $\beta = -1/2$, $\alpha \geqslant 1/2$. This is in part for technical reasons and in part because the discussion of such a case will become clearer after proving the theorem in the next section.

3. This section is devoted to the proof of the following theorem.

THEOREM 1. *Suppose $S \subseteq R^N$ is an open bounded set whose boundary has finite upper Minkowski measure. Let f denote a bounded complex-valued function on R^N, satisfying the assumptions* (3)–(7), *with $\beta \neq -1/2$ if $\alpha \geqslant 1/2$. Let F_R be defined as in* (2) *and γ as in* (10). *Set $p_f = 2N/N + 2(n+\gamma)$. Then for all $R > 0$*

(a) *if $n + \gamma \leqslant N/2$*

$$\|F_R\|_p \leqslant A_p R^{N/2-(n+\gamma)} \qquad \text{if } 1 \leqslant p < p_f, \tag{17}$$

$$\|F_R\|_p \leqslant A_p R^{N(p-1)/p} \log^{1/p} R \qquad \text{if } p = p_f, \tag{18}$$

$$\|F_R\|_p \leqslant A_p R^{N(p-1)/p} \qquad \text{if } p_f < p \leqslant 2, \tag{19}$$

where A_p is a constant not depending on R.

(b) *If $n + \gamma > N/2$*

$$\|F_R\|_1 \leqslant A \tag{20}$$

where A is a constant not depending on R.

PROOF. Choose a function $\psi \in C^\infty(R^N)$ with support contained in a ball D of radius $1/2$ centered at the origin, in such a way that

$$\int_{R^N} \psi(x)\, dx = 1, \qquad \int_{R^N} x^\xi \psi(x)\, dx = 0 \tag{21}$$

where ξ is a multi-index, $0 < |\xi| \leqslant n$. Set $f_R(x) = f(R^{-1}x)$. Then, following Shapiro **[8]**, we may write

$$\|F_R\|_p \leqslant \Big\| \sum_m ((f_R * \psi)(m) - f_R(m)) \exp(2\pi i m t) \Big\|_2 + \Big\| \sum_m (f_R * \psi)(m) \exp(2\pi i m t) \Big\|_p = \Sigma_2 + \Sigma_p. \tag{22}$$

In (22) $*$ denotes the convolution in R^N. We shall prove the theorem by producing the appropriate estimates for Σ_2 and Σ_p.

Estimates for Σ_2. Set $B_0 = \{x \in R^N\colon R^{-1}x \in S,\ d(R^{-1}x, \partial S) \geqslant 2R^{-1}\}$, $C_0 = S \setminus B_0$. Let $B = B_0 \cap Z^N$, $C = C_0 \cap Z^N$.

Then, using the first of (21) we get

$$\Sigma_2 \leqslant \left(\sum_{m \in B} \left| \int_D \{f(R^{-1}(m-x)) - f(R^{-1}m)\}\psi(x)\,dx \right|^2 \right)^{1/2}$$
$$+ \left(\sum_{m \in C} \left| \int_D f(R^{-1}(m-x))\psi(x)\,dx \right|^2 \right)^{1/2} + \left(\sum_{m \in C} |f(R^{-1}m)|^2 \right)^{1/2}$$
$$= I_1 + I_2 + I_3.$$

By the differentiability assumptions on f and the second equation of (21)

$$I_1 \leqslant R^{-(n+1)} \left(\sum_{m \in B} \left| \int_D \left\{ \sum_{|\xi| = n+1} (\xi!)^{-1} D^\xi f(R^{-1}(m + \theta x)) x^\xi \right\} \psi(x)\,dx \right|^2 \right)^{1/2}$$

where $\theta = \theta(x, m)$ is between 0 and 1. Let y_m denote a point in $\bar{D}$ such that $d(R^{-1}(m + x), \partial S)$ attains its maximum on $\bar{D}$. Then

$$I_1 \leqslant AR^{-(n+1)} \left(\sum_{m \in B} d(R^{-1}(m + y_m), \partial S)^{2\beta} \right)^{1/2}.$$

If $\beta \geqslant 0$, $I_1 \leqslant AR^{N/2-(n+1)} \leqslant AR^{N/2-(n+\gamma)}$, since B contains at most $AR^{N/2}$ points. If $\beta < 0$, we can reach the same result with little more effort. Namely, set $B_k = \{m \in B\colon R^{-1}(k-1) \leqslant d(R^{-1}m, \partial S) < R^{-1}k\}$ for $k = 3, 4, \ldots$. Clearly $B = \bigcup_k B_k$. It then follows (as $y_m \in \bar{D}$) that

$$I_1 \leqslant AR^{-(n+1)} \left(\sum_k \sum_{m \in B_k} (R^{-1}(k-2)(k+1)^{-1}(k+1))^{2\beta} \right)^{1/2}$$
$$\leqslant AR^{-(n+1)} \left(\sum_k \sum_{m \in B_k} (R^{-1}(k+1))^{2\beta} \right)^{1/2}.$$

But

$$(R^{-1}(k+1))^{2\beta} \leqslant \inf_{m \in B_k,\, x \in Q_N} d(R^{-1}(m+x), \partial S)^{2\beta}$$
$$\leqslant \inf_{m \in B_k} \int_{Q_N} d(R^{-1}(m+x), \partial S)^{2\beta}\,dx$$

so that, changing variables and performing the summations

$$I_1 \leqslant AR^{N/2-(n+1)} \left(\int_{S_{R^{-1}}} d(x, \partial S)^{2\beta}\,dx \right)^{1/2} \leqslant AR^{N/2-(n+\beta+3/2)}.$$

Here S_t has the same meaning as in Lemma 2 and the last inequality follows by the same arguments used in proving Lemma 2. The first inequality follows from the fact that $\bigcup_k \bigcup_{m \in B_k} R^{-1}(m + Q_N) \subseteq S_{R^{-1}}$. Therefore we have

$$I_1 \leqslant AR^{N/2-(n+\gamma)}. \tag{23}$$

To compute I_3, remark that C contains at most AR^{N-1} points, because of the assumption on ∂S. On the other hand, we have, by Lemma 1, that $|f(x)| \leqslant \mathrm{A}d(x, \partial S)^{\alpha+n}$ if $x \in S$. It follows that

$$I_3 \leqslant AR^{(N-1)/2} \sup_{m \in C} |f(R^{-1}m)| \leqslant AR^{(N-1)/2-(n+\alpha)} \leqslant AR^{N/2-(n+\gamma)}. \quad (24)$$

Analogously one shows

$$I_2 \leqslant AR^{N/2-(n+\gamma)}. \quad (25)$$

From (23)–(25) we have

$$\Sigma_2 \leqslant AR^{N/2-(n+\gamma)}. \quad (26)$$

Estimates for Σ_p. We may write the Poisson summation formula for $f_R * \psi$ (where $*$ denotes the convolution in R^N). We get

$$\sum_m f_R * \psi(m)\exp(2\pi imt) = \sum_m \hat{f}_R(m+t)\hat{\psi}(m+t) \quad \text{with } t \in T^N.$$

Therefore

$$\Sigma_p \leqslant \sup_m \|\hat{f}_R(m+t)\|_p \sum_m \|\hat{\psi}(m+t)\|_\infty \leqslant A \sup_m \|\hat{f}_R(m+t)\|_p \quad (27)$$

by the smoothness of ψ.

Suppose first $m \neq 0$. Then

$$\begin{aligned} \|\hat{f}_R(m+t)\|_p &= R^N \left\{ \int_{Q_N} |\hat{f}(R(m+t))|^p \, dt \right\}^{1/p} \\ &= R^{N(p-1)/p} \left\{ \int_{R(m+Q_N)} |\hat{f}(x)|^p \, dx \right\}^{1/p} \\ &\leqslant R^{N/2} \left\{ \int_{|x|>R/2} |\hat{f}(x)|^2 \, dx \right\}^{1/2}. \end{aligned}$$

By Lemma 2 and the definition of Nikolskii classes, we see that f belongs to $H_2^{n+\gamma}(R^N)$. Denote by $E_{R/2}(f)$ the best approximation to f in $L^2(R^N)$ by means of spherical entire functions of exponential type $R/2$ (see [6, p. 119]). It is well known that $E_{R/2}(f) = \{\int_{|x|>R/2} |\hat{f}(x)|^2 \, dx\}^{1/2}$. By [6, 5.2.1] we then get $E_{R/2}(f) \leqslant AR^{-(n+\gamma)}$

In conclusion we have, for $m \neq 0$,

$$\|\hat{f}_R(m+t)\|_p \leqslant AR^{N/2-(n+\gamma)}. \quad (28)$$

Suppose now $m = 0$. This time we have to evaluate the norm

$$I = \left(\int_{Q_N} |\hat{f}_R(t)|^p \, dt \right)^{1/p} = R^{N(p-1)/p} \left(\int_{RQ_N} |\hat{f}(t)|^p \, dt \right)^{1/p}.$$

We use the same method as in Yudin's paper [12]. Denote by $D(r)$ the ball of radius r centered at 0. Set $G_0 = D(1)$, $G_k = D(2^k) \setminus D(2^{k-1})$, $k = 1, 2, \ldots, O(\log_2 R)$. Then, by Hölder's inequality

$$I \leqslant R^{N(p-1)/p}\left(\sum_k \int_{G_k} |\hat{f}(x)|^p \, dx\right)^{1/p}$$

$$\leqslant A_p R^{N(p-1)/p}\left\{\|f\|_{L^2} + \sum_k \left(\int_{|x|>2^{k-1}} |\hat{f}(x)|^2 \, dx\right)^{p/2} 2^{kN(2-p)/2}\right\}^{1/p}.$$

If we argue for each term of the summation as in the case $m \neq 0$, we obtain

$$\left(\int_{|x|>2^{k-1}} |\hat{f}(x)|^2 \, dx\right)^{p/2} \leqslant A_p 2^{-(k-1)(n+\gamma)p}$$

so that

$$I \leqslant A_p R^{N(p-1)/p}\left\{\sum_k 2^{k(N(2-p)/2-p(n+\gamma))}\right\}^{1/p}. \tag{29}$$

Accordingly as $p < p_f$ or $p \geqslant p_f$ the sums in (29) diverge or converge, giving rise to different estimates for I. Collecting these estimates, as well as those arising from (26)–(28), part (a) of the theorem follows. Part (b) is routine, but we report the proof for completeness. Since $f \in H_2^{n+\gamma}(R^N)$, f belongs to the Liouville space $L_2^{n+\gamma-\varepsilon}(R^N)$ for every (small) positive ε (see [6, p. 323 and following]). Set $h = (1 + |x|^2)^{n+\gamma-\varepsilon}$. Then $h\hat{f} \in L^2(R^N)$ and $h^{-1} \in L^2(R^N)$ if $n + \gamma - \varepsilon > N/2$. It follows that $\hat{f} \in L^1(R^N)$, and that f is the Fourier transform of a $L^1(R^N)$ function. The Poisson summation formula then implies our assertion.

REMARK 1. If $p > 2$, then an interpolation between 2 and ∞ shows that $\|F_R\|_p \leqslant A_p R^{N(p-1)/p}$.

REMARK 2. As remarked by Yudin [12], when $\bar{S}$ is a compact convex set of the kind studied by Herz [4], and f is the characteristic function of $\bar{S}$, the estimates of Theorem 1 follow directly from the estimates proved by Herz [4] for $\hat{f}$ (see Shapiro [8] for the case when $\bar{S}$ is the unit ball).

4. In this section we study the case $\beta = -1/2$, $\alpha \geqslant 1/2$, which has been so far excluded. The results for this case are proved with minor changes in the proofs of §§2 and 3. However, if $\beta = -1/2$ and $\alpha \geqslant 1/2$, we make the supplementary assumptions on f (which are satisfied, for instance, in the case of Bochner-Riesz kernels)

$$f \in C^{n+2}(S), \tag{30}$$

$$|D^{\xi}f(x)| \leqslant \mathrm{Ad}(x, \partial S)^{-3/2} \quad \text{for all } \xi, |\xi| = n + 2, \text{ and } x \in S. \tag{31}$$

The assumptions (30)–(31) are natural, in the sense that they are satisfied, as remarked, by Bochner-Riesz kernels, and simplify the proofs considerably. For every $u \in L^2(R^N)$ we denote by $\omega_{2,2}(u, t)$ the second modulus of continuity of u in $L^2(R^N)$, i.e.

$$\omega_{2,2}(u, t) = \sup_{|h|<t} \|\Delta_h^2 u\|_{L^2},$$

where Δ_h^2 denotes the second difference, $\Delta_h^2 u(x) = u(x + 2h) - 2u(x + h) + u(x)$. The following lemma is similar to Lemma 2.

LEMMA 3. *Suppose S is an open bounded set such that ∂S has finite upper Minkowski measure. Suppose f is a bounded complex valued function on R^N satisfying* (3)–(7) *with $\alpha \geqslant 1/2$ and $\beta = -1/2$. Suppose moreover that f satisfies* (30) *and* (31). *Then*

$$\omega_{2,2}(D^{\xi}f, t) \leqslant At,$$

for every ξ such that $|\xi| = n$.

PROOF. We already know, from Lemma 2, that $D^{\xi}f \in L^2(R^N)$. Moreover, for $|h| \leqslant t$

$$\|\Delta_h^2 D^{\xi}f\|_{L^2} \leqslant \|\chi_{S_{3t}}\Delta_h^2 D^{\xi}f\|_{L^2} + 4\|\chi_{S\setminus S_{3t}}D^{\xi}f\|_{L^2} = I_1 + 4I_2.$$

I_2 can be evaluated in the same way as in Lemma 2, so that $I_2 \leqslant A|h|^{\alpha+1/2} \leqslant A|h|$, since $\alpha \geqslant 1/2$. As for I_1 we observe, by twice applying the mean value theorem and using (31), that

$$|\Delta_h^2 D^{\xi}f(x)| \leqslant A|h|^2 d(x + \theta_x h, \partial S)^{3/2}, \qquad 0 < \theta_x < 2,$$

so that $I_1^2 \leqslant A|h|^4 \int_{S_{3t}} d(x + \theta_x h, \partial S)^{-3}\,dx$. Arguing as in Lemma 2 we get $I_1^2 \leqslant A|h|^4(1 + t^{-2}) \leqslant A(t^4 + t^2)$. Hence $\omega_{2,2}(D^{\xi}f, t) \leqslant At$.

REMARK 3. This lemma implies that $f \in H_2^{n+1}(R^N)$.

THEOREM 2. *Suppose S is a bounded open subset of R^N whose boundary has finite upper Minkowski measure. Let f denote a bounded complex valued function on R^N satisfying* (3)–(7) *with $\alpha \geqslant 1/2$ and $\beta = -1/2$. Suppose moreover that f satisfies* (30) *and* (31). *Let F_R be defined as in* (2). *Then, if $p_f = 2N/N + 2(n+1)$,*

(a) *if $n + 1 \leqslant N/2$*

$$\|F_R\|_p \leqslant A_p R^{N/2-(n+1)} \qquad \text{if } 1 \leqslant p < p_f, \tag{32}$$

$$\|F_R\|_p \leqslant A_p R^{N(p-1)/p}\log^{1/p}R \qquad \text{if } p = p_f, \tag{33}$$

$$\|F_R\|_p \leqslant A_p R^{N(p-1)/p} \qquad \text{if } p_f < p \leqslant 2, \tag{34}$$

where A_p is a constant not depending on R.

(b) *If $n + 1 > N/2$, $\|F_R\|_1 \leqslant A$.*

PROOF. Taking into account Lemma 3 and Remark 3, one can repeat the proof of Theorem 1 verbatim, with the following modifications: the function ψ must be chosen in such a way that the second of (21) holds for every multi-index ξ with $|\xi| \leqslant n + 1$. Derivatives of order $n + 1$ must be replaced by derivatives of order $n + 2$, the number β must be replaced by $-3/2$ and γ by 1.

REMARK 4. Theorems 1 and 2 are our main result. When F_R coincides with a Bochner-Riesz kernel K_R^{δ}, the upper estimates provided by (17)–(19) and by (32)–(34) coincide with the estimates of **[1]** if $\delta < (N-1)/2$, and if $\delta = (N-1)/2$ with the estimate of Stein **[9]**. This can be easily seen from Example 2 in §2. Since for Bochner-Riesz kernels such estimates are sharp (for $p = 1$), it follows that (17)–(19) and (32)–(34) are sharp (for $p = 1$) for the class of kernels we are dealing with (of course they may not be sharp for particular kernels, e.g. when f is the characteristic function of a rectangle).

REMARK 5. The set S is supposed open because of conditions (3)–(7). However if S is only supposed to be compact (and ∂S has finite upper Minkowski measure), and if f is the characteristic function of S, then the arguments of Lemma 2 and Theorem 1 can be repeated, and we obtain essentially the theorem of Yudin **[12]**.

5. In this section we give an application of the results contained in the preceding sections. Namely we show that the estimates for the L^p norms of F_R proved in Theorems 1 and 2 give rise to a convergence result for the means (1) if g is supposed to belong to certain spaces of differentiable functions. First of all we recall that the Lipschitz space $\Lambda(k, p, q, T^N)$, $k > 0$, $1 \leqslant p, q \leqslant \infty$, is defined as the space of all $g \in L^p(T^N)$ such that all the (generalized) derivatives $D^\xi g$ with $|\xi| = r$ ($r + \tau = k$, r a nonnegative integer, $(0 < \tau \leqslant 1)$) belong to $L^p(T^N)$ and the seminorms

$$\left(\int_{R^N} |h|^{-N-q\tau} \|\Delta_h^2(D^\xi g)\|_p^q \, dh\right)^{1/q}$$

are finite (cf. **[6]** and **[11]**).

If $q = \infty$, then the definition gives the space $H_p^k(T^N)$. In any case $\Lambda(k, p, q, T^N)$ is continuously imbedded in $H_p^k(T^N)$ (see e.g. **[11]**). The following theorem was inspired by Ilin's result on the uniform convergence of spectral decompositions (see **[1]**).

THEOREM 3. *Suppose $S \subseteq R^N$ is a bounded open set such that ∂S has finite upper Minkowski measure and that $0 \in S$. Let f denote a bounded function on R^N satisfying* (3)–(7) *and also* (30)–(31) *if $\beta = -1/2$ and $\alpha \geqslant 1/2$. Suppose moreover that $f \in C^\infty(D(\delta))$ for some ball centered at 0 of radius δ, and that $f(0) = 1$. Then, for every $g \in \Lambda(k, p, q, T^N)$, $F_R * g$ converges uniformly to g if the following conditions are satisfied*:

$$1 \leqslant p \leqslant \infty, \qquad kp > N, \qquad k \geqslant N/2 - (n + \gamma), \qquad q < \infty, \tag{35}$$

where γ is as in (10) *or $\gamma = 1$ if $\beta = -1/2$ and $\alpha \geqslant 1/2$.*

PROOF. The second of (35) insures that $\Lambda(k, p, q, T^N)$ is continuously imbedded in $C(T^N)$ (see e.g. **[11**, Theorem 9′]). Choose $\psi \in C^\infty(R^N)$ in such a way that $\psi(x) = 1$ if $|x| \leqslant \delta/2$, $\psi(x) = 0$ for $|x| \geqslant \delta$. Define

$$\Psi_R(t) = \sum_m \psi(R^{-1}m)\exp(2\pi imt), \qquad t \in T^N,$$

where m ranges over Z^N. If $g \in \Lambda(k, p, q, T^N)$, we have by Young's inequality

$$\|F_R * g\|_\infty \leqslant \|F_R * \Psi_R * g\|_\infty + \|F_R\|_{p'} \|\Psi_R * g - g\|_p \tag{36}$$

with p and p' conjugate exponents. The assumptions on f and ψ, and the same argument used to establish part (b) of Theorem 1, show that the L^1 norms of $F_R * \Psi_R$ are uniformly bounded (with respect to R). On the other hand (see **[6**, 5.3–5.5]) there exists a trigonometric polynomial P_R such that $\|P_R - g\|_p \leqslant AR^{-k}$

and $\Psi_R * P_R = P_R$ (because $\psi(x) = 1$ for $|x| \leqslant \delta/2$). It therefore follows

$$\|\Psi_R * g - g\|_p \leqslant \|\Psi_R * (P_R - g)\|_p + \|P_R - g\|_p \leqslant AR^{-k}$$

where A depends on g. Now, it is easy to see that, if (35) are satisfied, it follows from the estimates of Theorems 1 and 2 that the term on the left of (36) is not greater than a constant A (depending on g). By the uniform boundedness theorem and the density of the trigonometric polynomials in the space $\Lambda(k, p, q, T^N)$ (see [**11**, Theorem 11′]), if $q < \infty$, the thesis follows from the fact that $f(0) = 1$.

REMARK 6. The condition $f \in C^\infty(D(\delta))$ can be of course weakened, e.g. requiring that $f \in H_2^h(D(\delta))$ with $h > N/2$.

REMARK 7. The conditions (35) are formally the same as in Ilin's paper [**1**]. It is quite clear that we could have obtained an analogous theorem e.g. in fractionary Sobolev spaces.

REFERENCES

1. S. A. Alimov, V. A. Ilin and E. M. Nikishin, *Convergence problems of multiple trigonometric series and spectral decomposition*. I, II, Uspehi Mat. Nauk **31** (1976), 28–83; **32** (1977), 107–130; English transl., Russian Math. Surveys **31** (1976), 29–86; **32** (1977), 115–139.

2. ______, *Conditions for the convergence of spectral decompositions that correspond to elliptic operators*. V. *A theorem on the uniform convergence of the spectral decomposition for a general second order elliptic operator*, Differencial′nye Uravnenija **10** (1974), 481–506; English transl., Differential Equations **10** (1974), 360–377.

3. S. A. Alimov, *Uniform convergence and summability of the spectral decomposition of functions in* L_p^q, Differencial′nye Uravnenija **9** (1973), 669–681; English transl., Differential Equations **9** (1973), 509–517.

4. C. Herz, *Fourier transforms related to convex sets*, Ann. of Math. (2) **75** (1962), 81–92.

5. V. A. Ilin, *Problems of localization and convergence for Fourier series in fundamental systems of the Laplace operator*, Uspehi Mat. Nauk **23** (1968); English transl., Russian Math. Surveys **23** (1968), 59–116.

6. S. M. Nikolskii, *Approximation of functions of several variables and imbedding theorems*, Springer-Verlag, Berlin, Heidelberg and New York, 1975.

7. H. S. Shapiro, *Topics in approximation theory*, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

8. ______, *Lebesgue constants for spherical partial sums*, J. Approx. Theory **13** (1975), 40–44.

9. E. M. Stein, *On certain exponential sums arising in multiple Fourier series*, Ann. of Math. (2) **73** (1961), 87–109.

10. ______, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

11. M. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space*, J. Math. Mech. **13** (1964), 407–479.

12. V. A. Yudin, *Behavior of Lebesgue constants*, Mat. Zametki **17** (1975); English transl., Math. Notes **17** (1975); 233–235.

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